

Partial extensions of jets and the polar distribution on Grassmannians of non-maximal integral elements

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Abstract

We study an intrinsic distribution, called *polar*, on the space of l -dimensional integral elements of the higher order contact structure on jet spaces. The main result establishes that this exterior differential system is the prolongation of a natural system of PDEs, named *pasting conditions*, on sections of the bundle of *partial jet extensions*. Informally, a partial jet extension is a k^{th} order jet with additional $k + 1^{\text{st}}$ order information along l of the n possible directions. A choice of partial extensions of a jet into all possible l -directions satisfies the pasting conditions if the extensions coincide along pairwise intersecting l -directions.

We further show that prolonging the polar distribution once more yields the space of (l, n) -dimensional integral flags with its double fibration distribution. When $l > 1$ the exterior differential system is holonomic, stabilizing after one further prolongation.

The proof starts from the space of integral flags, constructing the tower of prolongations by reduction.

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0. Introduction

Consider a manifold J supplied with an exterior differential system and let $\theta \in J$ be a point. The space I_θ^l of l -dimensional regular integral elements of the exterior differential system at θ (we refer to [5] for basic notions on exterior differential systems) is equipped with a natural distribution (in the sense of *field of tangent planes*): a tangent vector at $L \in I_\theta^l$ belongs to this distribution if, considered as an infinitesimal first order motion of the integral element L , it leaves L inside of its polar space. The existence of this distribution was pointed out to me by A. M. Vinogradov [10] and was called the *polar distribution* in [3].

In the special case when the exterior differential system is the higher order contact structure (a.k.a. Cartan distribution) on the manifold $J = J^k(2, 1)$ of k^{th} order jets of functions in two independent and one dependent variables, the polar distribution on the space of one-dimensional horizontal integral was shown to be locally isomorphic to a Cartan distribution on the jet space of a *new* bundle with *one* independent and $k + 1$ dependent variables [2]. This result was extended in [3] to the case of one-dimensional integral elements of contact manifolds, i.e. the case $J = J^1(n, 1)$. Both proofs were in local coordinates and gave no hint on the geometrical origin of this new bundle, nor on ways to extend the result to other dimensions.

Here I remedy this by:

- a) generalising the result to horizontal integral elements of arbitrary dimension $l < n$ in the Cartan distribution at a jet $\theta \in J^k(n, m)$, with an arbitrary number of independent and dependent variables n resp. m .
- b) giving a coordinate free description of this new bundle and clarifying its geometrical meaning.

The total space of the bundle mentioned in b) will be called the space of *partial jet extensions* of θ_k and denoted by $J_{\theta_k, l}^{k+1}$. An element of $J_{\theta_k, l}^{k+1}$ can be thought of as an extension of the k^{th} order jet θ_k by $k + 1^{\text{st}}$ order information in the direction of an l -dimensional subspace of the space of independent variables. In terms of local coordinates this means that an element of $J_{\theta_k, l}^{k+1}$ specifies the values of the partial derivatives of order $k + 1$, in l chosen directions, in addition to the partial derivatives of order $\leq k$ determined by θ_k . The base of the bundle of partial jet extensions is the Grassmannian $\text{Gr}(l, n)$ of all possible “directions” along which to extend. Hence a section of the bundle $J_{\theta_k, l}^{k+1} \rightarrow \text{Gr}(l, n)$ specifies a partial extension of θ_k along each l -dimensional direction. There is a natural condition for such a section to be “holonomic”: when all extensions coincide on intersecting directions. These *pasting conditions* can be reformulated as a system of first order linear PDEs on sections of $J_{\theta_k, l}^{k+1}$. We then have

Theorem 1 (Main result, first part). *The polar distribution on $I_{\theta_k}^l$, considered as an exterior differential system, is the $k - 1^{\text{st}}$ prolongation of the system of pasting conditions on the bundle of partial jet extensions.*

In the case of 1-dimensional integral elements the pasting conditions are trivially satisfied for any section, since there are no non-trivial intersections

of one dimensional directions. Hence in that case, by theorem 1, $I_{\theta_k}^l$ is the k^{th} jet space of the bundle $J_{\theta_k, l}^{k+1} \rightarrow \text{Gr}(l, n)$, in agreement with the results mentioned above [2, 3].

We further show

Theorem 2 (Main result, continued). *Prolonging the polar distribution once more leads to the space $I_{\theta_k}^{l, n}$ of (l, n) -dimensional integral flags with its canonical distribution induced from its double fibration structure. Finally, when $l > 1$, prolonging once more stabilizes the process leading to an involutive distribution whose integral leaves are in one to one correspond with “full” extensions of θ_k , i.e. jets of order $k + 1$.*

The proofs of these theorems proceed from the top of the prolongation tower down: we consider the space of integral flags and exhibit certain natural distributions on it. From these we construct the tower of prolongations by reduction. Along the way we introduce moving frames adapted to the distributions which allow explicitly computations of commutation relations.

Structure of the article

The article consist of four sections. The first one gives detailed definitions and a precise statement of the main result, while the remaining three sections contain the proof. In section 2 the tower of fibrations $I_{\theta_k}^{l, n} \rightarrow M^k \rightarrow \dots \rightarrow M^0 \rightarrow \text{Gr}(l, n)$ is constructed and M^k and M^0 are identified as $I_{\theta_k}^l$ and $J_{\theta_k, l}^{k+1}$. In section 3 we supply each manifold in the tower with a distribution and show that on M^k this coincides with the polar distribution. In the final section 4 we show that these distributions are consecutive prolongations and identify the one on M^1 with the pasting conditions.

Motivations and relations to other work

Spaces of lower dimensional integral elements in the Cartan distribution appear at several places in the theory of PDEs. They are central to characteristics, Monge cones, geometric singularities of PDEs [11, 7] and boundary conditions [9]. They have been used to find differential contact invariants of certain classes of PDEs [1, 8]. Flags of integral elements appear in the context of the Cartan-Kähler theorem [5]. To the authors knowledge, the polar distribution made its first appearance in the literature in [2, 3], where the reader may find simple applications to the classification of a third order PDE. The author is unaware of any previous appearance of the bundle of partial jet extension and the pasting conditions.

1. Definitions and statement of main results

1.1. Conventions on jets

We work in the setting of jets of n -dimensional submanifolds in a fixed $n + m$ -dimensional ambient manifold E . The space of k^{th} order jets is denoted with $J^k = J^k(E, n)$. We think of such jets as *infinitesimal germs* of n -submanifolds in E . The reader not familiar with jets of submanifolds might as well think of the locally isomorphic space of jets of sections of a bundle with m -dimensional fibers (corresponding to m dependent variables) and n -dimensional base (corresponding to n independent variables). We fix throughout a jet $\theta_k \in J^k$ of order $k \geq 1$ and denote with \mathcal{C}_{θ_k} the plane of the Cartan distribution at θ_k . The terminology *Cartan distribution* and *higher contact structure* are used synonymously. Manifolds are real, although all arguments remain valid over any field of characteristic 0.

We shall make use of several facts concerning jets and the Cartan distribution which we collect in this subsection. The initiated reader may want to skip ahead to subsection 1.2 and return here when necessary. Further notational conventions may be found in section 5.

For $k > r$ there are natural projections $\pi_{k,r} : J^k \rightarrow J^r$ forgetting higher order information of jets. We say that θ_k *extends* the jet $\theta_r \in J^r$ (or that θ_k *restricts* to θ_r) when $\pi_{k,r}(\theta_k) = \theta_r$. In particular, the restriction of θ_k to order 0 is a point in $E = J^0$ denoted with θ_0 .

We use the convention of indexing the fiber of a bundle with its base point, and hence denote the manifold of all $k + 1^{\text{st}}$ order jets extending θ_k with $J_{\theta_k}^{k+1}$.

There is a natural bijection between n -dimensional horizontal integral planes $R \subset \mathcal{C}_{\theta_k}$ and jets of order $k + 1$ extending θ_k . Such integral planes are called R-planes in [4]. The R-plane corresponding to $\theta_{k+1} \in J_{\theta_k}^{k+1}$ is denoted with $R_{\theta_{k+1}}$. The R-plane in $J^0 = E$ corresponding to the 1^{st} order restriction of θ_k will be denoted with $\underline{R} \subset T_{\theta_0}E$.

The fiber $J_{\theta_k}^{k+1}$ is affine with underlying vector space $S^{k+1}\underline{R}^* \otimes N$ [7], where N is the *normal tangent space* $T_{\theta_0}E/\underline{R}$. We will interpret tensors in $S^{k+1}\underline{R}^* \otimes N$ as homogeneous polynomial maps from \underline{R} to N of degree $k + 1$.

For a distribution \mathcal{E} on a manifold M the *curvature form* is the skew-symmetric tensor

$$\Omega : \mathcal{E} \wedge \mathcal{E} \rightarrow [\mathcal{E}, \mathcal{E}] / \mathcal{E} \quad (1.1)$$

induced by the Lie bracket of sections of \mathcal{E} . Here $[\mathcal{E}, \mathcal{E}]$ denotes the *derived distribution* of \mathcal{E} , which is the distribution spanned by \mathcal{E} and Lie-brackets

of fields in \mathcal{E} . The curvature form of the Cartan distribution \mathcal{C} is called the *metasymplectic form*. One can show that there is a natural isomorphism $[\mathcal{C}, \mathcal{C}] / \mathcal{C} \Big|_{\theta_k} \cong S^{k-1} \underline{R}^* \otimes N$ so the metasymplectic form is considered of type

$$\Omega : \mathcal{C}_{\theta_k} \wedge \mathcal{C}_{\theta_k} \rightarrow S^{k-1} \underline{R}^* \otimes N. \quad (1.2)$$

In standard local coordinates x_i, u^j, u_σ^j on jet spaces J^k , where x_i are the independent variables, u^j are the dependent variables and u_σ^j are jet coordinates with $\sigma = (\sigma_1, \dots, \sigma_m) \in \mathbb{N}^m$ a multiindex of length $|\sigma| \leq k$, the metasymplectic structure acts as

$$\Omega(D_i, D_j) = 0 \quad (1.3)$$

$$\Omega\left(\partial_{u_\sigma^j}, \partial_{u_{\sigma'}^{j'}}\right) = 0 \quad (1.4)$$

$$\Omega\left(\partial_{u_\sigma^j}, D_i\right) = \partial_{u_{\sigma-1_i}^j}. \quad (1.5)$$

Here $D_i = \partial_{x_i} + \sum_{|\sigma| < k} u_{\sigma+1_i}^j \partial_{u_\sigma^j}$ are *total derivatives* and the vertical fields $\partial_{u_\sigma^j}$ correspond to the homogeneous polynomials

$$\frac{1}{\sigma!} (dx_1)^{\sigma_1} \cdot \dots \cdot (dx_n)^{\sigma_n} \otimes \frac{\partial}{\partial u^i} \in S^{k-1} \underline{R}^* \otimes N \quad (1.6)$$

under the identification $[\mathcal{C}, \mathcal{C}] / \mathcal{C} \Big|_{\theta_k} \cong S^{k-1} \underline{R}^* \otimes N$.

1.2. Integral elements and the polar distribution

Recall that a vector subspace $L \subset \mathcal{C}_{\theta_k}$ is called an *integral element* [5] (or involutive subspace in [4]) of the Cartan distribution, if all differential forms in the differential ideal generated by the Cartan distribution vanish when restricted to L . Equivalently, L is integral if the metasymplectic form Ω vanishes when restricted to L . Such a plane is *horizontal* if it is transversal to fibers of the projection $\pi_{k,k-1} : J^k \rightarrow J^{k-1}$, which turns out to imply transversality with respect to $\pi_{k,0} : J^k \rightarrow J^0$.

Definition 1. The space of *horizontal l -dimensional integral elements of \mathcal{C}_{θ_k}* is

$$I_{\theta_k}^l := \{L \subset \mathcal{C}_{\theta_k} \mid \dim L = l, \Omega|_L = 0, L \text{ horizontal}\}. \quad (1.7)$$

Horizontal integral elements of maximal dimension are precisely R-planes [4].

To define the polar distribution on $I_{\theta_k}^l$ recall that the *polar space* L^\perp [5] of an integral element L is defined as the Ω -orthogonal of L :

$$L^\perp := \{v \in \mathcal{C}_{\theta_k} \mid \Omega(v, w) = 0 \ \forall w \in L\}. \quad (1.8)$$

Since there is a canonical identification

$$T_L \text{Gr}(\mathcal{C}_{\theta_k}, l) \cong \text{Hom}(L, \mathcal{C}_{\theta_k}/L) \quad (1.9)$$

[6] and since $I_{\theta_k}^l \subset \text{Gr}(\mathcal{C}_{\theta_k}, l)$, a tangent vector \dot{L} at $L \in I_{\theta_k}^l$ may be understood as a linear map

$$L \xrightarrow{\dot{L}} \frac{\mathcal{C}_{\theta_k}}{L}. \quad (1.10)$$

We define the *osculator* of \dot{L} as

$$\text{osc } \dot{L} := \text{pr}^{-1} \text{im } \dot{L} \quad (1.11)$$

where $\text{pr} : \mathcal{C}_{\theta_k} \rightarrow \frac{\mathcal{C}_{\theta_k}}{L}$ is the canonical projection and $\text{im } \dot{L}$ is the image of \dot{L} . The osculator may be thought of as the span of L and all infinitesimally near $L_t \in \text{Gr}(\mathcal{C}_{\theta_k}, l)$ reached by the infinitesimal displacement \dot{L} .

Using these notions we give

Definition 2. The plane of the *polar distribution* \mathcal{P} on $I_{\theta_k}^l$ at $L \in I_{\theta_k}^l$ is

$$\mathcal{P}_L := \left\{ \dot{L} \in T_L(I_{\theta_k}^l) \mid \text{osc } \dot{L} \subseteq L^\perp \right\}. \quad (1.12)$$

Alternatively we have the simpler but equivalent description

$$\mathcal{P}_L = \left\{ \dot{L} \in \text{Hom}(L, \mathcal{C}_{\theta_k}/L) \mid \Omega(l_1, \dot{L}(l_2)) = 0 \text{ for all } l_1, l_2 \in L \right\}, \quad (1.13)$$

where we interpret \dot{L} as the map 1.10 and $\dot{L}(l_2)$ is its application to l_2 . Description 1.13 follows from

Lemma 1. A vector $\dot{L} \in T_L \text{Gr}(\mathcal{C}_{\theta_k}, l)$ is tangent to the submanifold $I_{\theta_k}^l \subset \text{Gr}(\mathcal{C}_{\theta_k}, l)$ at $L \in I_{\theta_k}^l$ iff

$$\Omega(l_1, \dot{L}(l_2)) = \Omega(l_2, \dot{L}(l_1)) \quad \text{for all } l_1, l_2 \in L. \quad (1.14)$$

Proof. Given $\dot{L} \in T_L I_{\theta_k}^l$ we first show that 1.14 holds. For this consider a smooth one parameter family of planes $L_t \in I_{\theta_k}^l$ with $L_0 = L$, $\frac{d}{dt} L_t \Big|_{t=0} = \dot{L}$ and families of vectors $l_i(t) \in L_t$ with $l_i(0) = l_i$ and

$$\left(\frac{d}{dt} l_i(t) \Big|_{t=0} \mod L \right) = \dot{L}(l_i) \quad (1.15)$$

for $i = 1, 2$. Since L_t is integral we have

$$\Omega(l_1(t), l_2(t)) = 0. \quad (1.16)$$

Taking the derivative with respect to t at $t = 0$ on both sides of equation 1.16 using the product rule and 1.15 we obtain

$$\Omega(l_1, \dot{L}(l_2)) + \Omega(\dot{L}(l_1), l_2) = 0, \quad (1.17)$$

which by skew-symmetry of Ω leads to 1.14.

Conversely, assume $\dot{L} \in \text{Hom}(L, \mathcal{C}_{\theta_k}/L)$ satisfies 1.14 with $L \in I_{\theta_k}^l$. We need to show $\dot{L} \in T_L I_{\theta_k}^l$. Choose an R-plane R such that $L \subset R$ and choose a splitting $R = L \oplus L^{\text{comp}}$. This gives a splitting of the Cartan plane into three components $\mathcal{C}_{\theta_k} = L \oplus L^{\text{comp}} \oplus (S^k R^* \otimes N)$ where the last component is the tangent space to the fiber of $\pi_{k,k-1} : J^k \rightarrow J^{k-1}$. This induces a decomposition $\dot{L} = \dot{L}_{\text{vert}} \oplus \dot{L}_{\text{hor}}$ into a vertical $\dot{L}_{\text{vert}} : L \rightarrow L^{\text{comp}}$ and horizontal component $\dot{L}_{\text{hor}} : L \rightarrow S^k R^* \otimes N$. Substituting \dot{L} with $\dot{L}_{\text{vert}} \oplus \dot{L}_{\text{hor}}$ in 1.14 we find that \dot{L}_{vert} satisfies $\Omega(l_1, \dot{L}_{\text{vert}}(l_2)) + \Omega(\dot{L}_{\text{vert}}(l_1), l_2) = 0$. This implies that the graph of \dot{L}_{vert} is an l -dimensional integral element in \mathcal{C}_{θ_k} . Pick an R-plane R' such that $\text{graph}(\dot{L}_{\text{vert}}) \subset R'$ and interpret this new R-plane as the graph of a linear map $A : R \rightarrow S^k R^* \otimes N$. Since R' is integral it follows that $\Omega(r_1, A(r_2)) + \Omega(A(r_1), r_2) = 0$ for all $r_i \in R$. Moreover by construction $A(l) = \dot{L}_{\text{vert}}(l)$ for all $l \in L$.

We now define a one parameter family of l -dim planes $L_t \in \text{Gr}(\mathcal{C}_{\theta_k}, l)$ as follows. Pick a basis b_1, \dots, b_l of L and define L_t as the span of the vectors $b_i(t) := b_i + t\dot{L}_{\text{hor}}(b_i) + t \cdot A(b_i + t\dot{L}_{\text{hor}}(b_i))$. It is easy to see that $L_0 = L$ and $\frac{d}{dt} L_t \Big|_{t=0} = \dot{L}$. We claim that all the L_t are integral elements, which would finish the proof. For this it suffices to show that $\Omega(b_i(t), b_j(t)) = 0$ which follows from a straightforward computation. \square

1.3. The bundle of partial jet extensions

In this subsection we define the space of partial jet extensions of a k^{th} order jet θ_k , together with its fibration over a standard Grassmannian. We do this by introducing an equivalence relation on all $k + 1^{\text{st}}$ order jets extending θ_k .

Let $\theta_{k+1}, \theta'_{k+1} \in J_{\theta_k}^{k+1}$ be two $k + 1^{\text{st}}$ order jets extending θ_k and fix $\underline{L} \in \text{Gr}(\underline{R}, l)$. We think of $\theta_{k+1}, \theta'_{k+1}$ as infinitesimal germs of submanifolds in E having contact of order k along θ_k , while \underline{L} is thought of as an l -dimensional direction inside these germs.

Using local coordinates the equivalence relation is defined as follows: choose splitting coordinates $x_1, \dots, x_n, u_1, \dots, u_m$ on E centered at θ_0 such that \underline{L} is spanned by $\partial_{x_1}, \dots, \partial_{x_l}$. Let

$$u_j = F_j(x_1, \dots, x_n) \quad (1.18)$$

and

$$u_j = G_j(x_1, \dots, x_n) \quad (1.19)$$

be two sets of locally defined functions with $j = 1, \dots, m$, such that θ_{k+1} (resp. θ'_{k+1}) is the $k + 1^{\text{st}}$ jet of 1.18 (resp. 1.19). Hence the jets θ_{k+1} and θ'_{k+1} are determined by all partial derivatives of F and G at 0 of order $\leq k + 1$. Since θ_{k+1} and θ'_{k+1} are tangent of order k , all partial derivatives of F and G at 0 of order $\leq k$ are equal.

Definition 3. We say that θ_{k+1} and θ'_{k+1} are *tangent of order $k + 1$ in direction \underline{L}* if all $k + 1^{\text{st}}$ order partial derivatives of F and G involving *only* $\partial_{x_1}, \dots, \partial_{x_l}$ coincide.

An equivalent coordinate independent description is given by

Lemma 2. Two jets θ_{k+1} and θ'_{k+1} extending θ_k are tangent of order $k + 1$ along \underline{L} iff the polynomial $\theta_{k+1} - \theta'_{k+1} \in S^{k+1} \underline{R}^* \otimes N$ vanishes when restricted to \underline{L} .

Proof. From the properties of the affine $S^{k+1} \underline{R}^* \otimes N$ -structure on $J_{\theta_k}^{k+1}$. See for instance [7]. \square

It follows immediately that tangency of order $k + 1$ along \underline{L} is an equivalence relation on jets of order $k + 1$ extending θ_k , which leads to

Definition 4. We denote with $J_{\theta_k, \underline{L}}^{k+1}$ the quotient set under this equivalence relation and call it the *space of partial extensions of θ_k along \underline{L}* .

We think of an element in $J_{\theta_k, \underline{L}}^{k+1}$ as the jet θ_k with additional $k+1^{\text{st}}$ order information in direction of \underline{L} .

By varying \underline{L} in $\text{Gr}(\underline{R}, l)$, the spaces $J_{\theta_k, \underline{L}}^{k+1}$ make up the fibers of a bundle which we denote with

$$\mathbf{dir} : J_{\theta_k, l}^{k+1} \rightarrow \text{Gr}(\underline{R}, l) \quad (1.20)$$

$$\phi \mapsto \underline{L} = \mathbf{dir}(\phi) \quad (1.21)$$

where the projection \mathbf{dir} maps a partial extension ϕ to its direction of extension \underline{L} . By definition, a section of the bundle \mathbf{dir} specifies a partial jet extension of θ_k along every direction $\underline{L} \in \text{Gr}(\underline{R}, l)$.

There is a natural “holonomicity” condition for such a section.

Definition 5. We say a section $s : \text{Gr}(\underline{R}, l) \rightarrow J_{\theta_k, l}^{k+1}$ of partial jet extensions satisfies the *pasting conditions* (or is *holonomic*), if for any two directions $\underline{L}, \underline{L}' \in \text{Gr}(\underline{R}, l)$ the partial extensions $s(\underline{L}), s(\underline{L}')$ coincide on the intersection $\underline{L} \cap \underline{L}'$. This means that, jets θ_{k+1} and θ'_{k+1} representing $s(\underline{L})$ resp. $s(\underline{L}')$ have contact of order $k+1$ along $\underline{L} \cap \underline{L}'$.

Using lemma 2 it is straightforward to check that this definition is independent of the choice of representatives θ_{k+1} and θ'_{k+1} . We call these the *pasting conditions* since they express when a section of partial extensions can be “glued together” to form a full $k+1^{\text{st}}$ -order extension of θ_k . This last statement will actually be a consequence of the main result.

1.4. Infinitesimal pasting conditions

The pasting conditions can be reformulated as a system of 1^{st} order PDEs on sections of \mathbf{dir} , which we call the *infinitesimal pasting conditions*. To write down this system of PDEs we introduce local coordinates that shall be used throughout the rest of this article.

On the base space $\text{Gr}(\underline{R}, l)$ we choose standard affine coordinates on Grassmannians: fix an element $\underline{L}_0 \in \text{Gr}(\underline{R}, l)$, choose a basis

$$y_1, \dots, y_d \in \underline{L}_0^\circ \quad (1.22)$$

of the annihilator \underline{L}_0° and complement it to a basis of \underline{R}^* with covectors

$$x_1, \dots, x_l \in \underline{R}^*. \quad (1.23)$$

Then for any plane $\underline{L} \in \text{Gr}(\underline{R}, l)$ transversal to the complement

$$\underline{L}_0^{\text{compl}} := \bigcap_{1 \leq j \leq l} \ker x_j \quad (1.24)$$

there are unique coefficients $A_{i,j}$ such that the covectors

$$y_i - \sum_{j=1}^l A_{i,j} x_j \quad \text{with} \quad i = 1, \dots, d \quad (1.25)$$

form a basis of the annihilator of \underline{L} and conversely any such choice of coefficients determines such a plane. Hence the $A_{i,j}$ serve as local coordinates on $\text{Gr}(\underline{R}, l)$.

Coordinates on fibers: By lemma 2 each fiber $J_{\theta_k, \underline{L}}^{k+1}$ of **dir** is an affine quotient of $J_{\theta_k}^{k+1}$ with underlying vector space $S^{k+1} \underline{L}^* \otimes N$. Since $J_{\theta_k}^{k+1}$ is affine over $S^{k+1} \underline{R}^* \otimes N$ we fix a jet

$$\theta_{k+1, \text{orig}} \in J_{\theta_k}^{k+1} \quad (1.26)$$

as the “origin” and identify $J_{\theta_k}^{k+1}$ with the vector space $S^{k+1} \underline{R}^* \otimes N$ and hence $J_{\theta_k, \underline{L}}^{k+1}$ with $S^{k+1} \underline{L}^* \otimes N$. Since each \underline{L} transversal to $\underline{L}_0^{\text{compl}}$ is further identified with \underline{L}_0 by the natural projection $\underline{R} = \underline{L}_0 \oplus \underline{L}_0^{\text{compl}} \rightarrow \underline{L}_0$, we obtain an identification of $S^{k+1} \underline{L}^* \otimes N$ with $S^{k+1} \underline{L}_0^* \otimes N$ and hence an identification $J_{\theta_k, \underline{L}}^{k+1} \cong S^{k+1} \underline{L}_0^* \otimes N$. So choosing a basis

$$e_1, \dots, e_m \quad (1.27)$$

of N , each point in $J_{\theta_k, l}^{k+1}$ above our chart on $\text{Gr}(\underline{R}, l)$ is specified by its base coordinates $A_{i,j}$ plus the coefficients v_λ^h of a homogeneous polynomial

$$\sum_{\lambda, h} v_\lambda^h x^\lambda \otimes e_h \quad (1.28)$$

where $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{N}^l$ denotes a multiindex of length $|\lambda| = k+1$ and $x^\lambda = x_1^{\lambda_1} \cdots x_l^{\lambda_l}$.

With these local coordinates

$$A_{i,j}, v_\lambda^h \quad (1.29)$$

on the total space of **dir**, a local section of **dir** is given by functions

$$v_\lambda^h(A) \quad (1.30)$$

where A is short for all the variables $A_{i,j}$. Such a section satisfies the non-infinitesimal pasting conditions from definition 5 iff, for any two planes $\underline{L}, \underline{L}' \in \text{Gr}(\underline{R}, l)$ with coordinates A, A' we have

$$\sum_{\lambda, h} v_\lambda^h(A) x^\lambda \otimes e_h = \sum_{\lambda, h} v_\lambda^h(A') x^\lambda \otimes e_h \quad (1.31)$$

whenever $x = (x_1, \dots, x_l)$ satisfies

$$\sum_j A_{i,j} x_j = \sum_j A'_{i,j} x_j \quad (1.32)$$

for all $i = 1, \dots, d$.

To derive the infinitesimal pasting conditions from 1.31, 1.32 we fix $\underline{L} \in \text{Gr}(\underline{R}, l)$ with coordinates A and consider two continuous perturbations of \underline{L} : one perturbation changing entry $A_{i,j}$ of matrix A to $A_{i,j} + t$, with t a perturbation parameter, and leaving the other entries fixed. The other perturbation changing entry $A_{i,j'}$ to $A_{i,j'} + s$ with parameter s and leaving all other entries unperturbed. Here i, j, j' are fixed indices. We write the perturbed matrices as

$$A + t1_{i,j} \quad (1.33)$$

$$A + s1_{i,j'}. \quad (1.34)$$

For a section of **dir** that satisfies the pasting conditions, we substitute A with 1.33 and A' with 1.34 in 1.31 to obtain

$$\sum_\lambda v_\lambda^h(A + t1_{i,j}) x^\lambda \otimes e_h = \sum_\lambda v_\lambda^h(A + s1_{i,j'}) x^\lambda \otimes e_h \quad (1.35)$$

whenever $x = (x_1, \dots, x_l)$ satisfies

$$tx_j = sx_{j'} \quad (1.36)$$

according to 1.32. Taking the total differential of both sides of equations 1.35 and 1.36 (where the variables are s, t, x while A is assumed fixed, i.e.

$dA = 0$) we obtain

$$\begin{aligned} \sum_{\lambda} \partial_{A_{i,j}} v_{\lambda}^h(A + t1_{i,j}) x^{\lambda} dt \otimes e_h + \sum_{\lambda, \iota} v_{\lambda}^h(A + t1_{i,j}) \lambda_{\iota} x^{\lambda-1_{\iota}} dx_{\iota} \otimes e_h = \\ \sum_{\lambda} \partial_{A_{i,j'}} v_{\lambda}^h(A + s1_{i,j'}) x^{\lambda} ds \otimes e_h + \sum_{\lambda, \iota} v_{\lambda}^h(A + s1_{i,j'}) \lambda_{\iota} x^{\lambda-1_{\iota}} dx_{\iota} \otimes e_h \end{aligned} \quad (1.37)$$

from 1.35, while from 1.36 we obtain

$$x_j dt + t dx_j = x_{j'} ds + s dx_{j'}. \quad (1.38)$$

Now set $t = s = 0$, so 1.35 1.36 are trivially satisfied while 1.37 becomes

$$\sum_{\lambda} \partial_{A_{i,j}} v_{\lambda}^h(A) x^{\lambda} dt \otimes e_h = \sum_{\lambda} \partial_{A_{i,j'}} v_{\lambda}^h(A) x^{\lambda} ds \otimes e_h \quad (1.39)$$

after canceling equal terms. Equation 1.38 becomes

$$x_j dt = x_{j'} ds. \quad (1.40)$$

We may multiply both sides of equation 1.39 with x_j and substitute $x_j dt$ with $x_{j'} ds$ by 1.40 to find

$$\sum_{\lambda} \partial_{A_{i,j}} v_{\lambda}^h(A) x^{\lambda+1_{j'}} \otimes e_h = \sum_{\lambda} \partial_{A_{i,j'}} v_{\lambda}^h(A) x^{\lambda+1_j} \otimes e_h \quad (1.41)$$

where we have canceled ds . Since equations 1.41 hold for arbitrary values of x we can equate the coefficients on both sides to find that, in local coordinates, a section $v_{\lambda}^h(A)$ that satisfies pasting conditions 1.31, 1.32, also satisfies the *infinitesimal pasting conditions*

$$\partial_{A_{i,j}} v_{\lambda}^h = \partial_{A_{i,j'}} v_{\lambda'}^h \quad \text{whenever} \quad \lambda - 1_j = \lambda' - 1_{j'}, \quad (1.42)$$

$$\partial_{A_{i,j}} v_{\lambda}^h = 0 \quad \text{whenever} \quad \lambda_j = 0. \quad (1.43)$$

Observe that when $l = 1$ these conditions are trivially satisfied, so the equations are “empty”.

Remark 1. If one considers perturbations $A + t1_{i,j}$, $A + s1_{i',j'}$ with different indices $i \neq i'$ one finds again equations 1.43. In fact, the prolongation theorems 1 and 2 will establish that all possible differential consequences of the non-infinitesimal pasting conditions 1.31, 1.32 coincide with the differential consequences of the infinitesimal pasting conditions 1.42, 1.43.

1.5. Integral flags and the double fibration structure

A further ingredient of the main theorem is the space of partial flags of integral elements which we introduce here. This space is also the starting point for constructing the tower of prolongations and thereby for proving the main theorem.

Definition 6. A pair (L, R) of subspaces $L \subset R \subset \mathcal{C}_{\theta_k}$ with R an R-plane and L of dimension l will be called a (l, n) -dimensional flag of horizontal integral elements. The space of all such integral flags is denoted with

$$I_{\theta_k}^{l,n} := \left\{ (L, R) \mid L \in I_{\theta_k}^l, R \in I_{\theta_k}^n, L \subset R \right\}. \quad (1.44)$$

Remark 2. By established terminology it would be correct to call these *partial* flags. We omit the adjective to simplify the terminology.

The space of integral flags is naturally fibered in two ways: one projection forgets the smaller integral element L and remebers only R . Since R is an R-plane corresponding to some jet of order $k+1$ we write this projection as:

$$\mathbf{pr}_n : I_{\theta_k}^{l,n} \rightarrow J_{\theta_k}^{k+1} \quad (1.45)$$

$$(L, R_{\theta_{k+1}}) \mapsto \theta_{k+1}. \quad (1.46)$$

The second projection forgets R and is hence of the form

$$\mathbf{pr}_l : I_{\theta_k}^{l,n} \rightarrow I_{\theta_k}^l \quad (1.47)$$

$$(L, R_{\theta_{k+1}}) \mapsto L. \quad (1.48)$$

We picture both of these as a double fibration

$$\begin{array}{ccc} & I_{\theta_k}^{l,n} & \\ \mathbf{pr}_n \swarrow & & \searrow \mathbf{pr}_l \\ J_{\theta_k}^{k+1} & & I_{\theta_k}^l. \end{array} \quad (1.49)$$

This double fibration gives rise to a natural distribution on $I_{\theta_k}^{l,n}$. Recall that for a fiber bundle $\pi : A \rightarrow B$ the *vertical distribution* $V\pi$ on the total space A consists of all vectors tangent to the fibers.

Definition 7. The sum of the two vertical distributions associated to the projections \mathbf{pr}_l and \mathbf{pr}_n defines a distribution

$$\mathcal{F} := V\mathbf{pr}_l + V\mathbf{pr}_n \quad (1.50)$$

on $I_{\theta_k}^{l,n}$ which we call the *flag distribution* on the space of integral flags.

1.6. Statement of the main results

Before we state the main result we recall the notion of *prolongation of an exterior differential system with independence conditions*. We shall only need the case where the independence conditions are given by transversality conditions with respect to a bundle projection $\pi : M \rightarrow N$, and where the exterior differential system on M is a distribution \mathcal{E} (i.e. a Pfaffian system). We refer to [5] for the general definition.

To proceed we need to recall the notions of relative distribution and lift of a (relative) distribution.

Definition 8. Given a fiber bundle $f : A \rightarrow B$, a *relative distribution* \mathcal{D} *along* f is vector sub-bundle of the pullback of the tangent bundle TB to A . In other words, a relative distribution attaches to every point $a \in A$ a tangent plane $\mathcal{D}_a \subset T_{f(a)}B$ in a smooth way. Any relative distribution \mathcal{D} along f , can be *lifted* to a non-relative distribution $f^{-1}\mathcal{D}$ on A by defining $(f^{-1}\mathcal{D})_a := (Tf)^{-1}(\mathcal{D}_a)$.

Remark 3. Lifting relative distributions induces a canonical correspondence between relative distributions along f and distributions on A containing the vertical distribution Vf . Note also that every non-relative distribution on B can be seen as a relative distribution along f in an obvious way.

Returning to the notion of prolongation of a distribution (M, \mathcal{E}) , one defines the manifold $M^{(1)}$ to consist of all $(\dim N)$ -dimensional π -horizontal integral elements of \mathcal{E} . The prolonged distribution $\mathcal{E}^{(1)}$ on $M^{(1)}$ is then defined to be the lift of the *tautological relative distribution* along the natural projection $\pi^{(1)} : M^{(1)} \rightarrow M$. The tautological relative distribution by definition attaches to each $S \in M^{(1)}$ the subspace $S \subset T_{\pi^{(1)}(S)}M$. Since $M^{(1)}$ is still a bundle over N via $\pi \circ \pi^{(1)}$ we can iterate this construction and define the second prolongation etc.

Theorem 3 (Main theorem). *The $k - 1^{st}$ prolongation of the system of infinitesimal pasting conditions is the polar distribution on $I_{\theta_k}^l$. The k^{th} prolongation is the space of integral flags with its flag distribution. Moreover, when $l > 1$ the $k + 1^{st}$ prolongation is an involutive distribution whose maximal integral submanifolds are in one-to-one correspondence with jets of order $k + 1$ prolonging θ_k . When $l = 1$ the pasting conditions are empty and so $I_{\theta_k}^l = J^k(\mathbf{dir})$ and $I_{\theta_k}^{l,n} = J^{k+1}(\mathbf{dir})$ while the polar and flag distributions are the Cartan distributions on $J^k(\mathbf{dir})$ resp. $J^{k+1}(\mathbf{dir})$.*

Remark 4. In local coordinates an exterior differential system is a system of PDE's, while prolonging amounts to taking total derivatives of the equations and adding them to the system. For this reason we occasionally refer to prolongation as “adding differential consequences” to a system of PDEs.

2. Constructing the tower of fibrations

In this section we exhibit a natural chain of involutive distributions $\mathcal{V}^0 \subset \mathcal{V}^1 \subset \dots \subset \mathcal{V}^{k+2}$ on the space of integral flags $I_{\theta_k}^{l,n}$. Their leaf spaces then yield the tower of fiber bundles

$$I_{\theta_k}^{l,n} \rightarrow M^k \rightarrow \dots \rightarrow M^0 \rightarrow M^{-1}. \quad (2.1)$$

Having done that, we recognize M^k as $I_{\theta_k}^l$, M^0 as $J_{\theta_k, l}^{k+1}$ and M^{-1} as $\text{Gr}(\underline{R}, l)$. In section 3 we then show how each M^q , $q > 1$ is equipped with a natural distribution.

2.1. Internal structure of the tangent space $TI_{\theta_k}^{l,n}$

Since any integral element $L \in I_{\theta_k}^l$ is transversal to $\pi_{k,0} : J^k \rightarrow J^0$ we may project it down to $\underline{R} \subset T_{\theta_0} E$ to obtain a subspace we denote with $\underline{L} \in \text{Gr}(\underline{R}, l)$ (This projection also induces a canonical isomorphism $L \cong \underline{L}$ which we shall use implicitly). Hence $I_{\theta_k}^l$ is naturally fibered over $\text{Gr}(\underline{R}, l)$:

$$I_{\theta_k}^l \rightarrow \text{Gr}(\underline{R}, l) \quad (2.2)$$

$$L \mapsto \underline{L} = T\pi_{k,0}(L). \quad (2.3)$$

Using this projection we note the following important decomposition of the space of integral flags.

Lemma 3. *The map*

$$I_{\theta_k}^{l,n} \rightarrow \text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1} \quad (2.4)$$

$$(L, R_{\theta_{k+1}}) \mapsto (\underline{L}, \theta_{k+1}) \quad (2.5)$$

is a canonical diffeomorphism of manifolds.

Proof. The inverse can be described by

$$(\underline{L}, \theta_{k+1}) \mapsto \left(R_{\theta_{k+1}} \cap (T_{\theta_k} \pi_{k,0})^{-1}(\underline{L}), R_{\theta_{k+1}} \right). \quad (2.6)$$

□

We henceforth use this identification $I_{\theta_k}^{l,n} = \text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1}$ without explicit mention. It immediately exposes the following “internal structure” on tangent spaces of $I_{\theta_k}^{l,n}$.

Corollary 1. *The tangent space $T_{(L,R)} I_{\theta_k}^{l,n}$ at $(L, R) \in I_{\theta_k}^{l,n}$ is canonically isomorphic to*

$$\text{Hom}(\underline{L}, \underline{R}/\underline{L}) \oplus \left(S^{k+1} \underline{R}^* \otimes N \right). \quad (2.7)$$

Proof. The two summands correspond precisely to the tangent spaces of the components $\text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1}$. \square

A generic vector in $\text{Hom}(\underline{L}, \underline{R}/\underline{L}) \oplus (S^{k+1} \underline{R}^* \otimes N)$ will henceforth be denoted with $h \oplus f$, where $h \in \text{Hom}(\underline{L}, \underline{R}/\underline{L})$ and $f \in S^{k+1} \underline{R}^* \otimes N$.

2.1.1. A filtration on homogeneous polynomials

The subspace $\underline{L} \subset \underline{R}$ associated to an integral flag (L, R) gives rise to a filtration on the second component $S^{k+1} \underline{R}^* \otimes N$ of the tangent space of $I_{\theta_k}^{l,n}$ at (L, R) :

Definition 9. For $p = 0, 1, \dots, k+2$ define $U_{\underline{L}}^p$ to be the vector subspace of $S^{k+1} \underline{R}^* \otimes N$ consisting of all homogeneous polynomials that vanish after taking p derivatives in direction of \underline{L} . Equivalently, $U_{\underline{L}}^p$ consists of all symmetric $k+1$ -multilinear forms on \underline{R} that vanish when inserting p elements of \underline{L} .

These subspaces form a natural filtration in $S^{k+1} \underline{R}^* \otimes N$ depending on $\underline{L} \in \text{Gr}(\underline{R}, l)$:

$$\underbrace{U_{\underline{L}}^0}_{=0} \subset U_{\underline{L}}^1 \subset \dots \subset U_{\underline{L}}^{k+1} \subset \underbrace{U_{\underline{L}}^{k+2}}_{=S^{k+1} \underline{R}^* \otimes N}. \quad (2.8)$$

A basis of $U_{\underline{L}}^p$ may be constructed as follows: fix a basis y_1, \dots, y_d of \underline{L}° and complement it with forms x_1, \dots, x_l to a basis of \underline{R}^* . Denote symmetric monomials of these basic forms with

$$y^\delta x^\lambda := y_1^{\delta_1} \dots y_d^{\delta_d} x_1^{\lambda_1} \dots x_l^{\lambda_l} \quad (2.9)$$

where $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{N}^d$ and $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{N}^l$ are multi indices. Let e_1, \dots, e_m be the basis 1.27 of N .

Lemma 4. $U_{\underline{L}}^p$ is generated by all symmetric tensors $y^\delta x^\lambda \otimes e_h$ which are of degree less than p in the x 's. More formally

$$U_{\underline{L}}^p = \left\langle y^\delta x^\lambda \otimes e_h \mid |\delta| + |\lambda| = k + 1, |\lambda| < p, h = 1, \dots, m \right\rangle. \quad (2.10)$$

Proof. This follows straightforwardly from interpreting such symmetric tensors as polynomial maps. \square

Denoting with

$$q := k + 1 - p \quad (2.11)$$

the *complementary degree* to p , we may describe $U_{\underline{L}}^p$ as all symmetric tensors of degree at least q in the y 's.

Corollary 2.

$$U_{\underline{L}}^1 = S^{k+1} \underline{L}^\circ \otimes N \quad (2.12)$$

$$U_{\underline{L}}^{k+1} = \text{polynomials vanishing on } \underline{L} \quad (2.13)$$

2.2. Higher vertical distributions on $I_{\theta_k}^{l,n}$

The filtration 2.8 of $S^{k+1} \underline{R}^* \otimes N$ from the previous subsection induces a natural chain of distributions on the tangent spaces of $I_{\theta_k}^{l,n}$. From these we will construct the tower of prolongations 2.1.

Definition 10. For $p = 0, \dots, k + 2$ define the p^{th} vertical distribution \mathcal{V}^p on $I_{\theta_k}^{l,n}$ at a point $(L, R) \in I_{\theta_k}^{l,n}$ as

$$\mathcal{V}_{(L,R)}^p := \left\{ 0 \oplus f \in \text{Hom}(\underline{L}, \underline{R}/\underline{L}) \oplus \left(S^{k+1} \underline{R}^* \otimes N \right) = T_{(L,R)} I_{\theta_k}^{l,n} \mid f \in U_{\underline{L}}^p \right\}. \quad (2.14)$$

It is clear from 2.8 that

$$\underbrace{\mathcal{V}^0}_{=0} \subset \mathcal{V}^1 \subset \dots \subset \mathcal{V}^{k+2} \quad (2.15)$$

and the biggest vertical distribution \mathcal{V}^{k+2} is just the vertical distribution with respect to the projection $I_{\theta_k}^{l,n} \rightarrow \text{Gr}(\underline{R}, l)$. The terminology *vertical* distribution stems from the fact that we will quotient $I_{\theta_k}^{l,n}$ by these distributions to obtain the manifolds M^q in the tower $I_{\theta_k}^{l,n} \rightarrow M^k \rightarrow \dots \rightarrow M^0 \rightarrow M^{-1}$ and hence the \mathcal{V}^p are indeed vertical distributions.

The fact that we are allowed to quotient follows from the next

Lemma 5. *All higher vertical distributions \mathcal{V}^p are involutive, their integral leaves are affine spaces and their spaces of leaves are manifolds.*

Proof. The claim is clear for \mathcal{V}^{k+2} since this is the vertical distribution of the projection $I_{\theta_k}^{l,n} = \text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1} \rightarrow \text{Gr}(\underline{R}, l)$. To check the claim for the other vertical distributions note that, since each $\mathcal{V}^p \subset \mathcal{V}^{k+2}$, it suffices to verify it on each fiber of $I_{\theta_k}^{l,n} \rightarrow \text{Gr}(\underline{R}, l)$. But each fiber $I_{\theta_k}^{l,n} \big|_{\underline{L}}$ is the affine space $J_{\theta_k}^{k+1}$ and the distribution $\mathcal{V}^p \big|_{J_{\theta_k}^{k+1}}$ is a flat affine distribution there

$$\mathcal{V}^p \big|_{J_{\theta_k}^{k+1}} = U_{\underline{L}}^p \times J_{\theta_k}^{k+1} \subset \left(S^{k+1} \underline{R}^* \otimes N \right) \times J_{\theta_k}^{k+1} = T J_{\theta_k}^{k+1}. \quad (2.16)$$

Hence the integral leaves are parallel affine subspaces of $J_{\theta_k}^{k+1}$ modeled on the vector space $U_{\underline{L}}^p$ and the space of leaves is a smooth manifold. \square

Definition 11. The space of integral leaves of the distribution \mathcal{V}^p is denoted with M^q where $q = k + 1 - p$ is the complementary degree.

This way we get the tower of fiber bundles

$$M^{k+1} \rightarrow M^k \rightarrow \dots \rightarrow M^0 \rightarrow \underbrace{M^{-1}}_{=\text{Gr}(\underline{R}, l)} \quad (2.17)$$

where each M^q is a bundle over M^{q-1} with affine fibers.

2.3. Identifying $I_{\theta_k}^l$ and $J_{\theta_k, l}^{k+1}$ in the tower

It is clear that the highest component $M^{k+1} = I_{\theta_k}^{l,n}$ and that the lowest $M^{-1} = \text{Gr}(\underline{R}, l)$. The second highest M^k is $I_{\theta_k}^l$ by the next

Lemma 6. *The distribution \mathcal{V}^1 is the vertical distribution of the fibration $I_{\theta_k}^{l,n} \rightarrow I_{\theta_k}^l$, so $M^k = I_{\theta_k}^l$.*

Proof. If $(L, R_{\theta_{k+1}})$ and $(L, R_{\theta'_{k+1}})$ are in the same fiber of $\mathbf{pr}_l : I_{\theta_k}^{l,n} \rightarrow I_{\theta_k}^l$ then $\theta_{k+1} - \theta'_{k+1} \in S^{k+1} \underline{R}^* \otimes N$ is a polynomial vanishing when taking one derivative in direction of \underline{L} since $L \subset \left(R_{\theta_{k+1}} \cap R_{\theta'_{k+1}} \right)$, which is equivalent by definition to $\theta_{k+1} - \theta'_{k+1} \in U_{\underline{L}}^1$. \square

Next we have

Lemma 7. $M^0 = J_{\theta_k, l}^{k+1}$.

Proof. Two flags $(L, R_{\theta_{k+1}})$ and $(L', R_{\theta'_{k+1}})$ are in the same leaf of the distribution \mathcal{V}^{k+1} if and only if $\underline{L} = \underline{L}'$ and $\theta_{k+1} - \theta'_{k+1} \in U_{\underline{L}}^{k+1}$, but this last condition is precisely the condition that all $k+1^{\text{st}}$ derivatives of θ_{k+1} and θ'_{k+1} in direction of \underline{L} agree, hence they define the same partial jet prolongation of θ_k . \square

So we have identified the following components:

$$\underbrace{M^{k+1}}_{=I_{\theta_k}^{l,n}} \rightarrow \underbrace{M^k}_{=I_{\theta_k}^l} \rightarrow M^{k-1} \rightarrow \dots \rightarrow \underbrace{M^0}_{=J_{\theta_k, l}^{k+1}} \rightarrow \underbrace{M^{-1}}_{=\text{Gr}(\underline{R}, l)}. \quad (2.18)$$

3. Supplying the tower with distributions

Our next aim is to supply each M^q with a natural distribution $\underline{\mathcal{F}}^q$. We proceed by exhibiting a second chain of distributions on $I_{\theta_k}^{l,n}$ which will then descend to the M^q 's by a process of symmetry reduction.

3.1. Higher flag distributions on $I_{\theta_k}^{l,n}$

Definition 12. For $p = -1, 0, 1, \dots, k+1$ define the p^{th} flag distribution \mathcal{F}^p on $I_{\theta_k}^{l,n}$ as the sum of \mathcal{V}^{p+1} with the distribution vertical to the projection $\text{pr}_n : I_{\theta_k}^{l,n} \rightarrow J_{\theta_k}^{k+1}$.

So the plane of the p^{th} flag distribution at a point (L, R) is

$$\mathcal{F}_{(L, N)}^p = \left\{ h \oplus f \in \text{Hom}(\underline{L}, \underline{R}/\underline{L}) \oplus \left(S^{k+1} \underline{R}^* \otimes N \right) = T_{(L, R)} I_{\theta_k}^{l,n} \mid f \in U_{\underline{L}}^{p+1} \right\}. \quad (3.1)$$

It is clear that

$$\underbrace{\mathcal{F}^{-1}}_{=V \text{pr}_n} \subset \mathcal{F}^0 \subset \dots \subset \underbrace{\mathcal{F}^{k+1}}_{=T I_{\theta_k}^{l,n}}. \quad (3.2)$$

Concerning the second smallest distribution \mathcal{F}^0 we have

Lemma 8. \mathcal{F}^0 is the flag distribution \mathcal{F} of $I_{\theta_k}^{l,n}$.

Proof. This is a direct consequence of the definitions and lemma 6. \square

Remark 5. We shall see later (corollary 3), that the *higher flag* distributions are derived distributions (in the sense defined in subsection 1.1) of \mathcal{F}^0 . This, together with the previous lemma 8, justifies the terminology.

To explain how the distributions \mathcal{F}^p descend to M^q we recall the notion of characteristic symmetries of a distribution [4].

Definition 13. A vector field X is called a *characteristic symmetry* of a distribution \mathcal{E} , if it is contained in \mathcal{E} and a symmetry of \mathcal{E} (so Lie brackets of X with fields in the distribution remain in the distribution).

Characteristic symmetries form an involutive sub-distribution of \mathcal{E} and one may always locally quotient \mathcal{E} by the characteristic distribution to obtain a distribution on the space of integral leaves of the characteristic distribution. We call this process the *reduction of \mathcal{E} by characteristic symmetries*.

Hence, to proceed, our aim is to prove the following

Theorem 4. *For all $p = 0, \dots, k+1$ the characteristic distribution of \mathcal{F}^p is \mathcal{V}^p .*

To achieve this we construct an explicit basis of the higher flag distribution using local coordinates and compute its commutation relations in the following subsection.

3.2. Local coordinates, a non-holonomic frame and commutators

We start by introducing local coordinates on each component of the splitting $I_{\theta_k}^{l,n} = \text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1}$. Since we will later introduce a second set of local coordinates on $I_{\theta_k}^{l,n}$, adapted to the projections $I_{\theta_k}^{l,n} \rightarrow M^q$, we call this first set *trivial* and the second *adapted*.

3.2.1. Trivial local coordinates

As in subsection 1.4 we use affine coordinates $A_{i,j}$ on $\text{Gr}(\underline{R}, l)$ and identify the second component $J_{\theta_k}^{k+1}$ with the vector space $S^{k+1}\underline{R}^* \otimes N$ using the chosen “origin jet” 1.26.

Using the bases 1.22, 1.23 and 1.27 of subsection 1.4, a basis of $S^{k+1}\underline{R}^* \otimes N$ is given by divided powers

$$\frac{1}{\delta! \lambda!} y^\delta x^\lambda \otimes e_h \quad (3.3)$$

with $|\delta| + |\lambda| = k+1$. Here the factorial $\delta!$ of a multiindex is $\delta_1! \cdots \delta_d!$.

Definition 14. The dual basis to the divided powers 3.3 will be denoted with $u_{\delta,\lambda}^h$ and serves as local coordinates on $J_{\theta_k}^{k+1}$. The coordinates

$$A_{i,j}, u_{\delta,\lambda}^h \quad (3.4)$$

are called *trivial coordinates* on $I_{\theta_k}^{l,n}$.

3.2.2. A non-holonomic frame adapted to vertical distributions

Recall that a tangent vector at a point $(\underline{L}, R) \in I_{\theta_k}^{l,n}$ can be identified with an element $h \oplus f \in \text{Hom}(\underline{L}, \frac{R}{\underline{L}}) \oplus (S^{k+1}\underline{R}^* \otimes N)$. Such an $h \oplus f$ is in the distribution \mathcal{V}^p at the point (\underline{L}, R) iff $f \in U_{\underline{L}}^p$ and $h = 0$. Hence, according to lemma 4 and the definition of the coordinates $A_{i,j}$, the “partially” divided powers

$$\left(\frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda \right) \otimes e_h, \text{ with } |\lambda| < p \quad (3.5)$$

form a basis of \mathcal{V}^p at each point of $I_{\theta_k}^{l,n}$ (we have suppressed the component $h = 0$). In the previous equation the notation $(y - \sum Ax)^\delta$ stands for $(y_1 - \sum_j A_{1,j}x_j)^{\delta_1} \cdots (y_d - \sum_j A_{d,j}x_j)^{\delta_d}$.

Definition 15. Local vector fields on $I_{\theta_k}^{l,n}$ corresponding to the partially divided powers 3.5 will be denoted with $V_{\delta,\lambda}^h$ and called *vertical fields*.

These vertical fields $V_{\delta,\lambda}^h$ together with the coordinate fields $\partial_{A_{i,j}}$ clearly form a (non-holonomic) local frame on $I_{\theta_k}^{l,n}$.

The $V_{\delta,\lambda}^h$ will play an analogous role to the vertical coordinate fields $\partial_{u_{\sigma}^j}$ on jet spaces [4] while the $\partial_{A_{i,j}}$ will play an analogous role to the total derivatives D_i on jet spaces. For this reason and since we later introduce a second set of coordinates in which the current $\partial_{A_{i,j}}$ will have a different expansion, we adopt the following terminology.

Definition 16. The fields $\partial_{A_{i,j}}$ from the current chart will be denoted with $D_{i,j}$ and called *homogeneous total derivatives*.

Remark 6. The adjective *homogeneous* will be justified after comparing the commutation relations 3.8 and the expansion 4.10 of the $D_{i,j}$, with the analogous commutation relations and expansion of classical total derivatives D_i on jet spaces.

It is evident that the frame $D_{i,j}, V_{\delta,\lambda}^h$ is adapted to the higher vertical- and flag distributions in the sense that

$$\mathcal{V}^p = \left\langle V_{\delta,\lambda}^h \mid |\lambda| < p \right\rangle \quad (3.6)$$

$$\mathcal{F}^p = \left\langle D_{i,j}, V_{\delta,\lambda}^h \mid |\lambda| \leq p \right\rangle \quad (3.7)$$

To prove that the distributions \mathcal{V}^p are the characteristics of \mathcal{F}^p we compute the commutators of the frame.

Theorem 5. *All commutators of the frame $D_{i,j}, V_{\delta,\lambda}^h$ are zero except for the commutators $[V_{\delta,\lambda}^h, D_{i,j}]$ when $\delta_i > 0$. In that case we have:*

$$[V_{\delta,\lambda}^h, D_{i,j}] = V_{\delta-1_i, \lambda+1_j}^h. \quad (3.8)$$

Proof. That $[D_{i,j}, D_{i',j'}] = 0$ is clear since in the chosen coordinates these fields are just partial derivatives. That $[V_{\delta,\lambda}^h, V_{\delta',\lambda'}^{h'}] = 0$ is also easily seen, since by equation 3.5, the $V_{\delta,\lambda}^h$ are linear combinations of the coordinate fields $\partial_{u_{\delta,\lambda}^h}$ with coefficients depending only on the coordinates $A_{i,j}$.

We are left to consider the Lie brackets $[V_{\delta,\lambda}^h, D_{i,j}]$. We compute how these act on coordinate functions. First note that $[V_{\delta,\lambda}^h, D_{i,j}](A_{i',j'}) = 0$ since

$$[V_{\delta,\lambda}^h, D_{i,j}](A_{i',j'}) = V_{\delta,\lambda}^h(\underbrace{D_{i,j}(A_{i',j'})}_{=\text{constant}}) - D_{i,j}(\underbrace{V_{\delta,\lambda}^h(A_{i',j'})}_{=0}) = 0. \quad (3.9)$$

Now consider the action of $[V_{\delta,\lambda}^h, D_{i,j}]$ on a coordinate function $u_{\Delta,\Lambda}^H$ where $\Delta \in \mathbb{N}^d$ and $\Lambda \in \mathbb{N}^n$ are multi-indices and $H = 1, \dots, m$:

$$V_{\delta,\lambda}^h(\underbrace{D_{i,j}(u_{\Delta,\Lambda}^H)}_{=0}) - D_{i,j}(V_{\delta,\lambda}^h(u_{\Delta,\Lambda}^H)) = -D_{i,j}(V_{\delta,\lambda}^h(u_{\Delta,\Lambda}^H)). \quad (3.10)$$

To continue the computation consider the inner term $V_{\delta,\lambda}^h(u_{\Delta,\Lambda}^H)$ on the r.h.s. When $h \neq H$ this is obviously 0. In the case $h = H$ note that $V_{\delta,\lambda}^h(u_{\Delta,\Lambda}^h)$ is the

coefficient in front of $\partial_{u_{\Delta,\Lambda}^h}$ in the expansion of $V_{\delta,\lambda}^h$ in the coordinate frame. But this is the same as the coefficient in the expansion of $\frac{1}{\delta!}(y - \sum Ax)^\delta x^\lambda$ in front of $\frac{1}{\Delta! \Lambda!} y^\Delta x^\Lambda$. This coefficient may be computed by applying the operator

$$\partial_y^\Delta \partial_x^\Lambda := \partial_{y_1}^{\Delta_1} \dots \partial_{y_d}^{\Delta_d} \partial_{x_1}^{\Lambda_1} \dots \partial_{x_l}^{\Lambda_l} \quad (3.11)$$

to $\frac{1}{\delta!}(y + \sum Ax)^\delta x^\lambda$ since all polynomials involved are homogenous. So we have

$$V_{\delta,\lambda}^h(u_{\Delta,\Lambda}^h) = \partial_y^\Delta \partial_x^\Lambda \left(\frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda \otimes e_h \right). \quad (3.12)$$

Plugging this in in the r.h.s of equation 3.10 we obtain

$$\left[V_{\delta,\lambda}^h, D_{i,j} \right] (u_{\Delta,\Lambda}^h) = -\partial_{A_{i,j}} \partial_y^\Delta \partial_x^\Lambda \left(\frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda \otimes e_h \right). \quad (3.13)$$

Now we can exchange the order of derivatives on the r.h.s and derive first w.r.t. $\partial_{A_{i,j}}$. Using the chain rule we compute:

$$\partial_{A_{i,j}} \left(\frac{1}{\delta!} (y - \sum Ax)^\delta x^\lambda \right) = -\delta_i \cdot x_j \frac{1}{\delta!} (y - \sum Ax)^{\delta-1_i} x^\lambda \quad (3.14)$$

$$= \begin{cases} 0 & \text{if } \delta_i = 0 \\ -\frac{1}{(\delta-1_i)!} (y + \sum Ax)^{\delta-1_i} x^{\lambda+1_j} & \text{if } \delta_i > 0. \end{cases} \quad (3.15)$$

So we arrive at:

$$\left[V_{\delta,\lambda}^h, D_{i,j} \right] (u_{\Delta,\Lambda}^h) = \begin{cases} 0 & \text{if } \delta_i = 0 \\ \partial_y^\Delta \partial_x^\Lambda \left(\frac{1}{(\delta-1_i)!} (y + \sum Ax)^{\delta-1_i} x^{\lambda+1_j} \otimes e_h \right) & \text{if } \delta_i > 0. \end{cases} \quad (3.16)$$

From this we conclude that $\left[V_{\delta,\lambda}^h, D_{i,j} \right] = 0$ if $\delta_i = 0$ while in the case when $\delta_i > 0$ the r.h.s. of the last equation is precisely $V_{\delta-1_i, \lambda+1_j}^h(u_{\Delta,\Lambda}^h)$ by equation 3.12. \square

A remarkable direct consequence of the commutation relations 3.8, which we shall not need in the remainder, is

Corollary 3. *All flag distributions \mathcal{F}^p with $p \geq 1$ are derived distributions of the flag distribution $\mathcal{F} = \mathcal{F}^0$. More precisely*

$$\mathcal{F}^{p+1} = [\mathcal{F}^p \mathcal{F}^p] \quad (3.17)$$

for all $p = 0, \dots, k$.

3.3. The reduced distributions and identifying the polar distribution

Commutation relations 3.8 immediately imply theorem 4, hence the flag distribution \mathcal{F}^p reduces to a distribution on M^q for $q = k + 1 - p$ and $0 \leq q \leq k + 1$ by quotienting out characteristic symmetries.

Definition 17. The *reduction of the flag distribution \mathcal{F}^p to M^q* is denoted with $\underline{\mathcal{F}}^q$, where $q = k + 1 - p$ is the complementary degree to p .

Since $\mathcal{F}^{k+1} = TI_{\theta_k}^{l,n}$ we have $\underline{\mathcal{F}}^0 = TM^0$. Further since $\mathcal{V}^0 = 0$ and $\mathcal{F}^0 = \mathcal{F}$ by lemma 8, we have $\underline{\mathcal{F}}^{k+1} = \mathcal{F}$. So the tower 2.17 is now enhanced with distributions as follows:

$$\underbrace{(M^{k+1}, \underline{\mathcal{F}}^{k+1})}_{=(I_{\theta_k}^{l,n}, \mathcal{F})} \rightarrow (M^k, \underline{\mathcal{F}}^k) \rightarrow (M^{k-1}, \underline{\mathcal{F}}^{k-1}) \rightarrow \dots \rightarrow (M^0, \underbrace{\underline{\mathcal{F}}^0}_{=TM^0}) \rightarrow \text{Gr}(\underline{R}, l) \quad (3.18)$$

We already established $M^k = I_{\theta_k}^l$. We now claim that $\underline{\mathcal{F}}^k$ is the polar distribution \mathcal{P} . For this it suffices to prove the following

Proposition 1. \mathcal{F}^1 is the lift of the polar distribution \mathcal{P} from $I_{\theta_k}^l$ to $I_{\theta_k}^{l,n}$ via $I_{\theta_k}^{l,n} \rightarrow I_{\theta_k}^l$.

Proof. Fix $(L, R) \in I_{\theta_k}^{l,n}$. By the definition of \mathcal{F}^1 we need to show that for any tangent vector at $(L, R) \in I_{\theta_k}^{l,n}$ of the form $0 \oplus f \in \text{Hom}(\underline{L}, \underline{R}/\underline{L}) \oplus (S^{k+1}\underline{R}^* \otimes N)$ the following conditions are equivalent:

- 1) $f \in U_{\underline{L}}^2$
- 2) $T_{(L,R)}\mathbf{pr}_l(0 \oplus f) \in \mathcal{P}_L$

where $T_{(L,R)}\mathbf{pr}_l$ is the tangent map of $\mathbf{pr}_l : I_{\theta_k}^{l,n} \rightarrow I_{\theta_k}^l$ at (L, R) . Let

$$df : \underline{R} \rightarrow S^k \underline{R}^* \otimes N \quad (3.19)$$

denote the total differential of the polynomial $f \in S^{k+1}\underline{R}^* \otimes N$ and let

$$df \Big|_L : L \rightarrow \mathcal{C}_{\theta_k}/L \quad (3.20)$$

denote its restriction to L . In 3.20 we have implicitly used the canonical isomorphism $L \cong \underline{L}$ and the natural inclusion $S^k \underline{R}^* \otimes N \subset \mathcal{C}_{\theta_k}/L$ as vertical

tangent space to the projection $J^k \rightarrow J^{k-1}$. It is straightforward to see that for any $0 \oplus f \in T_{(L,R)} I_{\theta_k}^{l,n}$

$$T_{(L,R)} \mathbf{pr}_l(0 \oplus f) = \mathrm{d}f \Big|_L. \quad (3.21)$$

Now we compute with $l_1, l_2 \in L$

$$\Omega(l_1, T_{(L,R)} \mathbf{pr}_l(0 \oplus f)(l_2)) = \Omega\left(l_1, \mathrm{d}f \Big|_L(l_2)\right) \quad (3.22)$$

$$= \partial_{l_1} \partial_{l_2} f \quad (3.23)$$

where 3.23 follows from the structural properties of the metasymplectic form Ω 1.5. But 3.23 is zero for all $l_1, l_2 \in L \cong \underline{L}$ iff $f \in U_{\underline{L}}^2$ so the claim follows from description 1.13. \square

It remains to identify the pasting conditions in the tower. We will do this in the next section together with the proof that consecutive components of the tower are prolongations.

4. Proving that the tower prolongs the pasting conditions

4.1. Consecutive M^q 's are prolongations

Our next aim is to prove that each distributions $(M^q, \underline{\mathcal{F}}^q)$ is the prolongation of the previous $(M^{q-1}, \underline{\mathcal{F}}^{q-1})$ for $q > 1$. Denote the projection with

$$\Pi_{q,q-1} : M^q \rightarrow M^{q-1} \quad (4.1)$$

and let ϕ_q be a point in the fiber $M_{\phi_{q-1}}^q$ over $\phi_{q-1} \in M^{q-1}$. Attached to ϕ_q is the plane $\underline{\mathcal{F}}_{\phi_q}^q$ of the distribution $\underline{\mathcal{F}}^q$ which we may project down to M^{q-1} . We denote the projected plane with

$$Q_{\phi_q} := T_{\phi_q} \Pi_{q,q-1}(\underline{\mathcal{F}}_{\phi_q}^q). \quad (4.2)$$

These “Q-planes” are analogous to the R-planes in jets spaces by the following three results which together prove that each $(M^q, \underline{\mathcal{F}}^q)$ is the prolongation of $(M^{q-1}, \underline{\mathcal{F}}^{q-1})$ for $q > 1$.

Proposition 2. *For each $\phi_q \in M^q$ with $q = 1, \dots, k+1$, the plane Q_{ϕ_q} is a horizontal maximal integral element in $(M^{q-1}, \underline{\mathcal{F}}^{q-1})$ of dimension $\dim \mathrm{Gr}(\underline{R}, l)$. Horizontal here means transversal to $M^{q-1} \rightarrow M^{q-2}$, which turns out to be equivalent to being transversal to $M^{q-1} \rightarrow \mathrm{Gr}(\underline{R}, l)$.*

Proposition 3. *For all $q = 1, \dots, k + 1$, the map*

$$\phi_q \mapsto Q_{\phi_q} \quad (4.3)$$

is an injection from the fiber $M_{\phi_{q-1}}^q$ into the space of horizontal maximal integral elements of $\underline{\mathcal{F}}^{q-1}$ at ϕ_{q-1} .

So we may identify M^q with a subset of maximal horizontal integral elements of $(M^{q-1}, \underline{\mathcal{F}}^{q-1})$. In fact, for $q \geq 2$, *any* maximal integral elements of $(M^{q-1}, \underline{\mathcal{F}}^{q-1})$ is of the form Q_{ϕ_q} , which is the content of the next

Proposition 4. *For all $q = 2, \dots, k + 1$, the map*

$$\phi_q \mapsto Q_{\phi_q} \quad (4.4)$$

is a surjection from the fiber $M_{\phi_{q-1}}^q$ to horizontal maximal integral elements of $\underline{\mathcal{F}}^{q-1}$ at ϕ_{q-1} .

To prove propositions 2, 3, 4 we introduce a second set of coordinates on $I_{\theta_k}^{l,n}$ which descend to the quotients M^q . This allows us to give explicit bases of the reduced distributions $\underline{\mathcal{F}}^q$ and compute their commutation relations.

4.2. Local coordinates and non holonomic frames on the M^q 's

Since we fixed a jet $\theta_{k+1, \text{orig}} \in J_{\theta_k}^{k+1}$ in 1.26 to identify $J_{\theta_k}^{k+1}$ with the vector space $S^{k+1} \underline{R}^* \otimes N$, we may consider

$$\text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1} \rightarrow \text{Gr}(\underline{R}, l) \quad (4.5)$$

to be a vector bundle. The partially divided powers $\frac{1}{\delta!}(y - \sum Ax)^\delta x^\lambda \otimes e^h$ then form a basis in each fiber. This frame is “moving” from fiber to fiber as it depends on the base coordinates $A_{i,j}$. Here $A_{i,j}$ and x, y have the same meaning as in subsection 3.2.1.

Definition 18. The fiber-wise dual one-forms to the frame

$$\frac{1}{\delta!}(y - \sum Ax)^\delta x^\lambda \otimes e_h \quad (4.6)$$

will be denoted with $v_{\delta, \lambda}^h$ and provide new coordinates on the fibers of $\text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1} \rightarrow \text{Gr}(\underline{R}, l)$. Together with the coordinates $A_{i,j}$ on the base $\text{Gr}(\underline{R}, l)$ they constitute another set of local coordinates on $I_{\theta_k}^{l,n}$ which we call *adapted*.

Observe that in these adapted coordinates the vector fields $V_{\delta,\lambda}^h$ are just the partial derivatives $\partial_{v_{\delta,\lambda}^h}$

$$V_{\delta,\lambda}^h = \partial_{v_{\delta,\lambda}^h} \quad (4.7)$$

while the fields $D_{i,j}$ are *no longer* the coordinate fields $\partial_{A_{i,j}}$, as in the trivial coordinates.

It is clear from 4.7 that the coordinates $A_{i,j}, v_{\delta,\lambda}^h$ with $|\delta| \leq q$ descend to coordinates on M^q .

Our next aim is to expand the fields $D_{i,j}$ in the coordinates $A_{i,j}, v_{\delta,\lambda}^h$.

Proposition 5. *We have*

$$D_{i,j}(A_{i',j'}) = \begin{cases} 1 & \text{if } i = i' \text{ and } j = j' \\ 0 & \text{else} \end{cases} \quad (4.8)$$

$$D_{i,j}(v_{\delta,\lambda}^h) = \begin{cases} v_{\delta+1_i, \lambda-1_j}^h & \text{if } \lambda_j > 0 \\ 0 & \text{else} \end{cases}, \quad (4.9)$$

from which the coordinate expansion

$$D_{i,j} = \partial_{A_{i,j}} + \sum_{\lambda_j > 0} v_{\delta+1_i, \lambda-1_j}^h \partial_{v_{\delta,\lambda}^h} \quad (4.10)$$

follows. The sum on the r.h.s. of 4.10 runs over all repeated indices h, δ, λ .

Proof. Equation 4.8 is obvious if we recall that in the previous trivial coordinates the derivations $D_{i,j}$ were just the partial derivative with respect to $A_{i,j}$.

To prove the second equation 4.9 we first express the $u_{\delta,\lambda}^h$ and $v_{\delta,\lambda}^h$ as sections of the dual $S^{k+1}\underline{R} \otimes N^*$ using the dual basis to $y_1, \dots, y_d, x_1, \dots, x_l \in \underline{R}^*$ and $e_1^*, \dots, e_m^* \in N^*$ and the natural isomorphism

$$S^{k+1}(\underline{R}^*) \cong (S^{k+1}\underline{R})^* \quad (4.11)$$

induced from the non-degenerate pairing

$$S^{k+1}\underline{R} \otimes S^{k+1}(\underline{R}^*) \rightarrow \mathbb{R} \quad (4.12)$$

given by

$$w_1 \cdot \dots \cdot w_{k+1} \otimes \alpha_1 \cdot \dots \cdot \alpha_{k+1} \mapsto \sum_{\varsigma} \prod_{i=1}^{k+1} \langle w_{\varsigma(i)}, \alpha_i \rangle \quad (4.13)$$

where ς runs through all permutations of the set $\{1, \dots, k+1\}$. If r_1, \dots, r_n is a basis of \underline{R} and the associated dual basis of \underline{R}^* is denoted with r_1^*, \dots, r_n^* , then under identification 4.11 the dual basis of $r^\sigma \in S^{k+1}\underline{R}$ is mapped to $\frac{1}{\sigma!}(r^*)^\sigma \in S^{k+1}\underline{R}^*$.

So letting $y_1^*, \dots, y_d^*, x_1^*, \dots, x_l^* \in \underline{R}$ denote the basis dual to $y_1, \dots, y_d, x_1, \dots, x_l \in \underline{R}^*$ and $e_1^*, \dots, e_m^* \in N^*$ the one dual to $e_1, \dots, e_m \in N$, we have

$$u_{\delta, \lambda}^h = (y^*)^\delta (x^*)^\lambda \otimes e_h^*. \quad (4.14)$$

Further, since the basis of \underline{R} dual to the basis

$$\left(y_1 - \sum A_{1,j}x_j\right), \dots, \left(y_d - \sum A_{d,j}x_j\right), x_1, \dots, x_l \quad (4.15)$$

of R^* is given by

$$y_1^*, \dots, y_d^*, \left(x_1^* + \sum A_{i,1}y_i^*\right), \dots, \left(x_l^* + \sum A_{i,l}y_i^*\right) \quad (4.16)$$

we have

$$v_{\delta, \lambda}^h = \frac{1}{\lambda!} (y^*)^\delta (x^* + Ay^*)^\lambda \otimes e^{*h} \quad (4.17)$$

again by 4.11 and since the $v_{\delta, \lambda}^h$ are by definition dual to the basis $\lambda! \frac{1}{\delta! \lambda!} (y - \sum Ax)^\delta x^\lambda$. By expanding the powers on the r.h.s. of 4.17 we could express the coordinates $v_{\delta, \lambda}^h$ as linear combinations of the $u_{\delta, \lambda}$ with coefficients depending on the variables $A_{i,j}$. We shall not do this, instead we recall again that in the coordinates $u_{\delta, \lambda}$, $A_{i,j}$ the derivations $D_{i,j}$ act as partial derivative with respect to $A_{i,j}$. Hence applying the chain rule we can compute

$$D_{i,j}(v_{\delta, \lambda}^h) = \frac{\partial}{\partial A_{i,j}} \left(\frac{1}{\lambda!} (y^*)^\delta (x^* + Ay^*)^\lambda \otimes e_h^* \right) \quad (4.18)$$

$$= \lambda_j y_i^* \frac{1}{\lambda!} (y^*)^\delta (x^* + Ay^*)^{\lambda-1_j} \otimes e_h^* \quad (4.19)$$

$$= \begin{cases} \frac{1}{(\lambda-1_j)!} (y^*)^{\delta+1_i} (x^* + Ay^*)^{\lambda-1_j} \otimes e_h^* & \text{if } \lambda_j > 0 \\ 0 & \text{else} \end{cases} \quad (4.20)$$

$$= \begin{cases} v_{\delta+1_i, \lambda-1_j}^h & \text{if } \lambda_j > 0 \\ 0 & \text{else} \end{cases}. \quad (4.21)$$

□

Definition 19. The q -truncated homogeneous total derivatives are the vector fields on M^q (where $q \geq 0$) defined in local adapted coordinates by

$$D_{i,j}^{[q]} := \partial_{A_{i,j}} + \sum_{\substack{|\delta| \leq q \\ \lambda_j > 0}} v_{\delta+1_i, \lambda-1_j}^h \partial_{v_{\delta, \lambda}^h}. \quad (4.22)$$

It is clear that $D_{i,j}^{[k+1]} = D_{i,j}$.

Lemma 9. a) The fields $\{\partial_{v_{\delta, \lambda}^h}, D_{i,j}^{[q]}\}$ with $|\delta| \leq q$ make up a frame on M^q .

b) Commutators of this frame are all zero except for the commutators

$$[\partial_{v_{\delta, \lambda}^h}, D_{i,j}^{[q]}] = \partial_{v_{\delta-1_i, \lambda+1_j}^h} \quad (4.23)$$

when $|\delta_i| > 0$.

c) The vertical distribution of $M^q \rightarrow M^{q-1}$ is spanned by $\partial_{v_{\delta, \lambda}^h}$ with $|\delta| = q$.

d) The fields $\{\partial_{v_{\delta, \lambda}^h}, D_{i,j}^{[q]}\}$ with $|\delta| = q$ form a local basis of $\underline{\mathcal{F}}^q$ and split it into vertical and horizontal part.

Proof. Straightforward from the definitions and the previous results. \square

Corollary 4. For $q = 0, \dots, k+1$ any plane $Q \subset \underline{\mathcal{F}}_{\phi_q}^q$ of maximal dimension and horizontal to $M^q \rightarrow M^{q-1}$ has a basis of the form

$$C_{i,j} := D_{i,j}^{[q]} + \sum_{|\delta|=q} C_{i,j,h}^{\delta, \lambda} \partial_{v_{\delta, \lambda}^h} \quad (4.24)$$

with unique coefficients $C_{i,j,h}^{\delta, \lambda}$. It is hence of dimension $\dim \text{Gr}(\underline{R}, l)$ and horizontal to the projection $M^q \rightarrow \text{Gr}(\underline{R}, l)$.

Definition 20. We denote the curvature form of $\underline{\mathcal{F}}^q$ with $\Omega^{[q]}$. We may compute with it directly by using commutators 4.23.

Lemma 10. For $q = 1, \dots, k+1$ a horizontal plane $Q \subset \underline{\mathcal{F}}_{\phi_q}^q$ of dimension $\text{Gr}(\underline{R}, l)$ is an integral element of $\underline{\mathcal{F}}^q$ if and only if the coefficients $C_{i,j,h}^{\delta, \lambda}$ of its basis 4.24 satisfy

$$C_{i,j,h}^{\delta, \lambda} = C_{i',j',h}^{\delta', \lambda'} \quad (4.25)$$

whenever the indices satisfy

$$\delta_{i'} > 0, \delta'_i > 0 \quad (4.26)$$

$$\delta - 1_{i'} = \delta' - 1_i \quad (4.27)$$

$$\lambda + 1_{j'} = \lambda' + 1_j \quad (4.28)$$

and condition

$$C_{i,j,h}^{\delta,\lambda} = 0 \quad \text{whenever} \quad \lambda_j = 0 \quad \text{and} \quad l > 1. \quad (4.29)$$

Proof. The plane Q is integral if and only if

$$\Omega^{[q]}(C_{i,j}, C_{i',j'}) = 0 \quad (4.30)$$

for all i, j, i', j' . Expanding the left hand side of 4.30 leads to

$$\sum_{\substack{|\delta|=q \\ \delta_{i'} > 0}} C_{i,j,h}^{\delta,\lambda} \partial_{v_{\delta-1_{i'}, \lambda+1_{j'}}^h} - \sum_{\substack{|\delta|=q \\ \delta_i > 0}} C_{i',j',h}^{\delta,\lambda} \partial_{v_{\delta-1_i, \lambda+1_j}^h} = 0. \quad (4.31)$$

Changing indices in the first sum to $\Delta = \delta - 1_{i'}$, $\Lambda = \lambda + 1_{j'}$ and in the second to $\Delta = \delta - 1_i$, $\Lambda = \lambda + 1_j$ transforms equation 4.31 to

$$\sum_{\substack{|\Delta|=q-1 \\ \Lambda_{j'} > 0}} C_{i,j,h}^{\Delta+1_{i'}, \Lambda-1_{j'}} \partial_{v_{\Delta, \Lambda}^h} - \sum_{\substack{|\Delta|=q-1 \\ \Lambda_j > 0}} C_{i',j',h}^{\Delta+1_i, \Lambda-1_j} \partial_{v_{\Delta, \Lambda}^h} = 0. \quad (4.32)$$

Collecting bases we find

$$\begin{aligned} & \sum_{\substack{|\Delta|=q-1 \\ \Lambda_{j'} > 0 \\ \Lambda_j > 0}} \left(C_{i,j,h}^{\Delta+1_{i'}, \Lambda-1_{j'}} - C_{i',j',h}^{\Delta+1_i, \Lambda-1_j} \right) \partial_{v_{\Delta, \Lambda}^h} + \\ & + \sum_{\substack{|\Delta|=q-1 \\ \Lambda_{j'} > 0 \\ \Lambda_j = 0}} C_{i,j,h}^{\Delta+1_{i'}, \Lambda-1_{j'}} \partial_{v_{\Delta, \Lambda}^h} + \sum_{\substack{|\Delta|=q-1 \\ \Lambda_j > 0 \\ \Lambda_{j'} = 0}} C_{i',j',h}^{\Delta+1_i, \Lambda-1_j} \partial_{v_{\Delta, \Lambda}^h} = 0. \end{aligned} \quad (4.33)$$

Equating coefficients to zero and returning to the previous indices we find conditions 4.25 from the first summand of 4.33, while from the second and third summands (which are only present when $l > 1$) we find condition 4.29. \square

Lemma 11. For $q = 1, \dots, k+1$ a horizontal plane $Q \subset \underline{\mathcal{F}}_{\phi_{q-1}}^{q-1}$ is of the form Q_{ϕ_q} for some $\phi_q \in M_{\phi_{q-1}}^q$ if and only if the coefficients $C_{i,j,h}^{\delta,\lambda}$ of its basis 4.24 satisfy

$$C_{i,j,h}^{\delta,\lambda} = C_{i',j',h}^{\delta',\lambda'} \quad (4.34)$$

whenever the indices satisfy

$$\lambda_j > 0, \lambda'_{j'} > 0 \quad (4.35)$$

$$\delta + 1_i = \delta' + 1_{i'} \quad (4.36)$$

$$\lambda - 1_j = \lambda' - 1_{j'} \quad (4.37)$$

and condition

$$C_{i,j,h}^{\delta,\lambda} = 0 \quad \text{whenever} \quad \lambda_j = 0. \quad (4.38)$$

Proof. We start by showing that the basis of a plane Q_{ϕ_q} satisfies 4.34 and 4.38. By lemma 9 the plane $\underline{\mathcal{F}}_{\phi_q}^q$ is spanned by the fields $D_{i,j}^{[q]}$ and vertical fields $\partial_{v_{\delta,\lambda}^h}$ with $|\delta| = q$. The vertical ones are annihilated when projecting to M^{q-1} while the $D_{i,j}^{[q]}$ are mapped to

$$C_{i,j} := D_{i,j}^{[q-1]} + \sum_{\substack{|\delta|=q-1 \\ \lambda_j > 0}} v_{\delta+1_i, \lambda-1_j}^h \partial_{v_{\delta,\lambda}^h} \quad (4.39)$$

where now the numbers $v_{\delta+1_i, \lambda-1_j}^h$ on the r.h.s of 4.39 are to be understood as the coordinates of the point ϕ_q in the fiber over ϕ_{q-1} . Vectors 4.39 are a basis of Q_{ϕ_q} of the form 4.24 with $C_{i,j,h}^{\delta,\lambda} = v_{\delta+1_i, \lambda-1_j}^h$. It is straightforward to see that these coefficients satisfy 4.34 and 4.38.

Conversely suppose the basis $C_{i,j}$ of a plane $Q \subset \underline{\mathcal{F}}_{\phi_{q-1}}^{q-1}$ satisfies conditions 4.34 and 4.38. We need to find a point $\phi_q \in M_{\phi_{q-1}}^q$ such that $Q = Q_{\phi_q}$. For any multiindex $(\Delta, \Lambda) \in \mathbb{N}^d \times \mathbb{N}^l$ with $|\Delta| = q$, $|\Delta| + |\Lambda| = k+1$ and any $h \in 1, \dots, m$ define the numbers

$$v_{\Delta, \Lambda}^h := C_{i,j,h}^{\Delta-1_i, \Lambda+1_j} \quad (4.40)$$

where we choose i in such a way that $\Delta_i > 0$, which is always possible since $|\Delta| \geq 1$. By 4.34 this definition is independent of the choices of i, j . By further taking into consideration condition 4.38 we see that

$$C_{i,j} = D_{i,j}^{[q-1]} + \sum_{\substack{|\delta|=q-1 \\ \lambda_j > 0}} v_{\delta+1_i, \lambda-1_j}^h \partial_{v_{\delta,\lambda}^h} \quad (4.41)$$

which by 4.39 proves that Q is of the form Q_{ϕ_q} with the point $\phi_q \in M_{\phi_{q-1}}^q$ determined by the fiber coordinates 4.40. \square

We are now in the position to easily prove propositions 2,3 and 4.

Proof of proposition 2. By lemma 11 the basis $C_{i,j}$ of Q_{ϕ_q} satisfies conditions 4.34 and 4.38, which for $q > 1$ are the same as conditions 4.25 and 4.29 of lemma 10, hence Q_{ϕ_q} is integral. When $q = 1$, Q_{ϕ_1} is integral since $\underline{\mathcal{F}}^0 = TM^0$. \square

Proof of proposition 3. If $\phi_q \neq \tilde{\phi}_q$ are two distinct points over ϕ_{q-1} there must be indices δ, λ, h such that the corresponding fiber coordinates of the points differ $v_{\delta,\lambda}^h \neq \tilde{v}_{\delta,\lambda}^h$. Since $q > 0$ there is an i such that $\delta_i \neq 0$. Then the coefficients in front of $\partial_{v_{\delta-1_i, \lambda+i_j}^h}$ in the bases 4.39 of Q_{ϕ_q} and $Q_{\tilde{\phi}_q}$ differ, hence $Q_{\phi_q} \neq Q_{\tilde{\phi}_q}$ by uniqueness of the bases $C_{i,j}$. \square

Proof of proposition 4. For the range of indices q under consideration conditions 4.25 and 4.29 of lemma 10 coincide with conditions 4.34 and 4.38 of lemma 11 hence an integral Q is of the form Q_{ϕ_q} . \square

4.3. Identifying the pasting conditions in the tower

Finally, the PDEs we called infinitesimal pasting conditions 1.42 and 1.43 are encoded in $(M^1, \underline{\mathcal{F}}^1)$ as follows.

Proposition 6. *The image of the map*

$$\phi_1 \mapsto Q_{\phi_1}, \quad (4.42)$$

understood in the obvious way as a subset of the first order jet space of the bundle $J_{\theta_k, l}^{k+1} \rightarrow \text{Gr}(\underline{R}, l)$, is precisely the zero set of the infinitesimal pasting conditions 1.42 and 1.43.

Proof. Observe first that coordinates $A_{i,j}, v_{\lambda}^h$ used in the description of the infinitesimal pasting conditions 1.42 and 1.42 are precisely the adapted coordinates $A_{i,j}, v_{0,\lambda}^h$ on M^0 (where now $\delta = 0$). Fix a point $\phi_0 \in M^0$. Any $\dim(\text{Gr}(\underline{R}, l))$ -dimensional horizontal plane $Q \subset T_{\phi_0} M^0$ is now of the form

$$C_{i,j} = \partial_{A_{i,j}} + \sum C_{i,j,h}^{0,\lambda} \partial_{v_{0,\lambda}^h} \quad (4.43)$$

with unique coefficients $C_{i,j,h}^{0,\lambda}$ which may be thought of as fiber coordinates $v_{0,\lambda,i,j}^h$ in the first jet bundle of **dir** corresponding to partial derivatives $\partial_{A_{i,j}} v_{0,\lambda}^h$. By lemma 11, Q is of the form Q_{ϕ_1} iff the coefficients $C_{i,j,h}^{0,\lambda}$ satisfy

$$C_{i,j,h}^{0,\lambda} = C_{i,j',h}^{0,\lambda'} \quad (4.44)$$

whenever the indices satisfy

$$\lambda_j > 0, \lambda'_{j'} > 0 \quad (4.45)$$

$$\lambda - 1_j = \lambda' - 1_{j'} \quad (4.46)$$

and condition

$$C_{i,j,h}^{0,\lambda} = 0 \quad \text{whenever} \quad \lambda_j = 0. \quad (4.47)$$

These are precisely the pasting conditions 1.42 and 1.42. \square

We finish the proof of the main theorem with

Lemma 12. *When $l > 1$ The only maximal integral elements of $(I_{\theta_k}^{l,n}, \mathcal{F})$ transversal to $I_{\theta_k}^{l,n} \rightarrow I_{\theta_k}^l$ are the vertical tangent spaces of the projection*

$$\mathbf{pr}_n : \text{Gr}(\underline{R}, l) \times J_{\theta_k}^{k+1} \rightarrow J_{\theta_k}^{k+1} \quad (4.48)$$

i.e. planes of the distribution \mathcal{F}^{-1} . So the maximal integral submanifolds of \mathcal{F} are the fibers of $I_{\theta_k}^{l,n} \rightarrow J_{\theta_k}^{k+1}$ and hence correspond bijectively to “full” jets of order $k+1$ extending θ_k . This proves that the prolongation of $(I_{\theta_k}^{l,n}, \mathcal{F})$ is $(I_{\theta_k}^{l,n}, V\mathbf{pr}_n)$ which is an involutive distribution.

Proof. Follows directly from lemma 10 equation 4.29 since in this case $\lambda = 0$. \square

This concludes the proof of main theorem 3.

5. Notational conventions

For a finite dimensional vector space W over a field \mathbb{K} , and $V \subset W$ a subspace we use the following conventions:

1. $\text{Gr}(W, l)$ denotes the Grassmannian of all l dimensional subspaces of W .

2. $S^k W$ denotes the k^{th} symmetric tensor product of W .
3. W^* denotes the dual $\text{hom}(W, \mathbb{K})$.
4. $V^\circ \subset W^*$ denotes the annihilator of V .
5. W/V denotes the quotient.
6. $\langle S \rangle$ denotes the span of the subset $S \subset W$

For manifolds M, N and a map $f : M \rightarrow N$ we use the conventions:

1. $Tf : TM \rightarrow TN$ denotes the tangent map.
2. $f^{-1}(S)$ denotes the preimage of subset $S \subset N$ under $f : M \rightarrow N$.
3. $M_q := f^{-1}(\{q\})$ denotes the fiber over $q \in N$ when $f : M \rightarrow N$ is a bundle.
4. An f -horizontal plane is a tangent subspace of M transversal to the fibers of f .
5. Vf denotes the vertical distribution of f when it is a fiber bundle.
6. For a chart x_1, \dots, x_n on N the associated coordinate fields are denoted with ∂_{x_i}

For a multinindex $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{N}^n$ and variables x_1, \dots, x_n :

1. $x^\delta = x_1^{\delta_1} \cdot \dots \cdot x_n^{\delta_n}$.
2. $\delta! = \delta_1! \cdot \dots \cdot \delta_n!$ is the factorial of the multiindex.
3. $|\delta| = \delta_1 + \dots + \delta_n$ denotes the length of the multiindex.
4. 1_j denotes the multiindex with all zero entries except for the entry at position j equaling 1.

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6. References

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